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HYBRID STOCHASTIC-SUBJECTIVE APPROACHES TO TREATING UNCERTAINTY
MILESTONE (N°:M2.1.C.3)

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Foreword

The work presented in this report was developed within the Integrated Project PAMINA: **P**erformance **A**ssessment **M**ethodologies **IN** **A**pplication to Guide the Development of the Safety Case. This project is part of the Sixth Framework Programme of the European Commission. It brings together 25 organisations from ten European countries and one EC Joint Research Centre in order to improve and harmonise methodologies and tools for demonstrating the safety of deep geological disposal of long-lived radioactive waste for different waste types, repository designs and geological environments. The results will be of interest to national waste management organisations, regulators and lay stakeholders.

The work is organised in four Research and Technology Development Components (RTDCs) and one additional component dealing with knowledge management and dissemination of knowledge:

- In RTDC 1 the aim is to evaluate the state of the art of methodologies and approaches needed for assessing the safety of deep geological disposal, on the basis of comprehensive review of international practice. This work includes the identification of any deficiencies in methods and tools.

- In RTDC 2 the aim is to establish a framework and methodology for the treatment of uncertainty during PA and safety case development. Guidance on, and examples of, good practice will be provided on the communication and treatment of different types of uncertainty, spatial variability, the development of probabilistic safety assessment tools, and techniques for sensitivity and uncertainty analysis.

- In RTDC 3 the aim is to develop methodologies and tools for integrated PA for various geological disposal concepts. This work includes the development of PA scenarios, of the PA approach to gas migration processes, of the PA approach to radionuclide source term modelling, and of safety and performance indicators.

- In RTDC 4 the aim is to conduct several benchmark exercises on specific processes, in which quantitative comparisons are made between approaches that rely on simplifying assumptions and models, and those that rely on complex models that take into account a more complete process conceptualization in space and time.

The work presented in this report was performed in the scope of RTDC 2.

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Hybrid Stochastic-Subjective Approaches to Treating Uncertainty

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1. Introduction

One of the most important steps in doing performance assessment (PA) is a problem structuring, i.e. developing a mathematical model for the physical situation at hand. The model solution is a function of the model parameters (Φ), and is conditional on the model assumptions (*M*) and on the modeler's current state of knowledge (*H*) [1]. Generally, there are uncertainties associated with the conditional model. In such a case the model solution $h(A \mid \Phi, M, H)$ should be expressed in the form of the conditional probability of the event of interest *A* that results from solving the conditional model.

Further, the model uncertainties cover a wide variety of uncertainties, they comprise of aleatory (or stochastic) uncertainties as well as of epistemic (or state-of-knowledge) uncertainties. The selection of a proper mathematical model for an individual particular uncertainty of the model is fundamental, especially for the interpretation of $h(A \mid \Phi, M, H)$, which may have no meaning as a (relative) frequency of occurrence of the event *A*. Rather, it may express a measure of the modeler's "degree of belief" in assessing the uncertainty of *A*.

The interpretation of probability has been the subject of scientific dispute for centuries. There have been selected three main categories of probability interpretations.

1. The **classical** interpretation of **probability**, which is identified with the classical works of Pierre Simon Laplace [3], defines the probability of an event *S* as the fraction of favorable cases among the equally possible cases. In its applications, the classical probability takes the form of combinatorial probability and thus provides the framework for the derivation of the standard probability distributions, like the binomial distribution (expresses e.g. the probability of observing *r* failures in *n* independent trials), the Poisson distribution (expresses e.g. the probability of *r* failures occurring in a fixed period of time), or the exponential distribution (models e.g. the survival time of a radioactive particle), etc.

2. The **frequentist** interpretation of **probability** determines the probability of an event *S* by its relative frequency of occurrence after repeating a process a large number of times under similar conditions. The central idea of frequentist probability is the sample with its statistical parameters. The statistical parameters like a sample mean or a sample variance correspond, respectively, to the first and the second moment of the probability distribution of a random variable. Ultimately, the parameters can be utilized to estimate the parameters of a probability distribution [6].

It is of course impossible to perform infinity of repetitions of a random experiment to determine the probability of an event. Rather, in practice, we are often restricted to only a limited number of repetitions of the process leading to different relative frequencies in different series of trials. If these relative frequencies are to define the probability, the probability will be slightly different every time it is measured. Thus, a frequentist estimate itself shall be assessed by using statistical measures like confidence intervals, tests of fit, etc. From a practical point of view, it is very difficult to propagate classical statistical confidence intervals through PA models to estimate a confidence interval for a result of interest [15].

3. **Subjective probability** is meant to be a measure of subjective confidence, or the degree to which the statement is supported by the available evidence. Although frequentist probabilities are purely associated with the events governed by some random physical phenomena, a subjective probability can be assigned to any statement, even when no random process is involved.

The mathematical model of uncertainty should reflect and incorporate the level of available information. PA often requires the investigation of the consequence of rare event for which only few data are available. In such a case it may be preferable to formalize uncertainty within the framework of the subjective theory of probability, by using one of the alternative ways that have been developed in the past decades.

Mathematical approaches to modeling uncertainty will be reviewed in the Section 2. Following the recent review [7], a broad spectrum of approaches ranging from interval analysis, probability theory, to fuzzy sets and possibility theory will be presented hierarchically, ordered from the perspective of set functions. The Section 3 is devoted to the transferable belief model of uncertainty. The Section 4 exemplifies the power of subjective approaches to treating uncertainty. SWOT analysis is presented in the section Discussion.

2. Mathematical approaches to the representation of uncertainty

The main mathematical approaches to the modeling of uncertainty will be described in this section using a problem of modeling the uncertainty of a single scalar variable. A variable will be denoted by upper case, e.g. *B*; while its realization by lower case *b*.

Interval analysis

The interval analysis is intended to estimate the bounds of the conditional model solutions. Any model uncertainty (input) is bounded within the interval $[b_L, b_U]$, determined by lower b_L and upper b_U limits (worst/best case assumption) without considering detailed information on a probability structure. The model evaluations result in a set of the model solutions, which could be in general of arbitrary geometry. A set of the model solutions is an interval, if the conditional model is continuous with respect to *B*. The basic operations of interval arithmetic are:

$$[b_L, b_U] + [c_L, c_U] = [b_L + c_L, b_U + c_U],$$

$$[b_L, b_U] - [c_L, c_U] = [b_L - c_U, b_U - c_L],$$

$$[b_L, b_U] \times [c_L, c_U] = [\min(b_L c_L, b_L c_U, b_U c_L, b_U c_U), \max(b_L c_L, b_L c_U, b_U c_L, b_U c_U)],$$

$$[b_L, c_U]/[c_L, c_U] = [\min(b_L / c_L, b_L / c_U, b_U / c_L, b_U / c_U), \max(b_L / c_L, b_L / c_U, b_U / c_L, b_U / c_U)]$$

.

The interval analysis is designed for cases when the probability structure of *B* is not known. However, if the probability structure of *B* is known, the method is not recommended. The input uncertainties are flattened out to the intervals ignoring thus available information.

Probability theory

The probability theory describes the uncertainties by means of the probability distributions $p_\lambda(b)$. The probability that the realizations of *B* lie in a set S is given by

$$P(B \in S) = \int_S p_\lambda(b) db .$$

The probability distribution $p_\lambda(b)$ can be viewed as a member of a class of distributions which is parameterized by parameters λ. Hence, the complete specification of a probability distribution requires determination of the class it belongs to as well as the values of its parameters. For example, the class of Weibull distributions

$$p_\lambda(b) = \frac{\alpha}{\beta}\left(\frac{b}{\beta}\right)^{\alpha-1} e^{-(b/\beta)^\alpha}, \ \alpha, \beta > 0, \ 0 \le b \le \infty,$$

is given by a scale parameter β and a shape parameter α (Fig. 1).

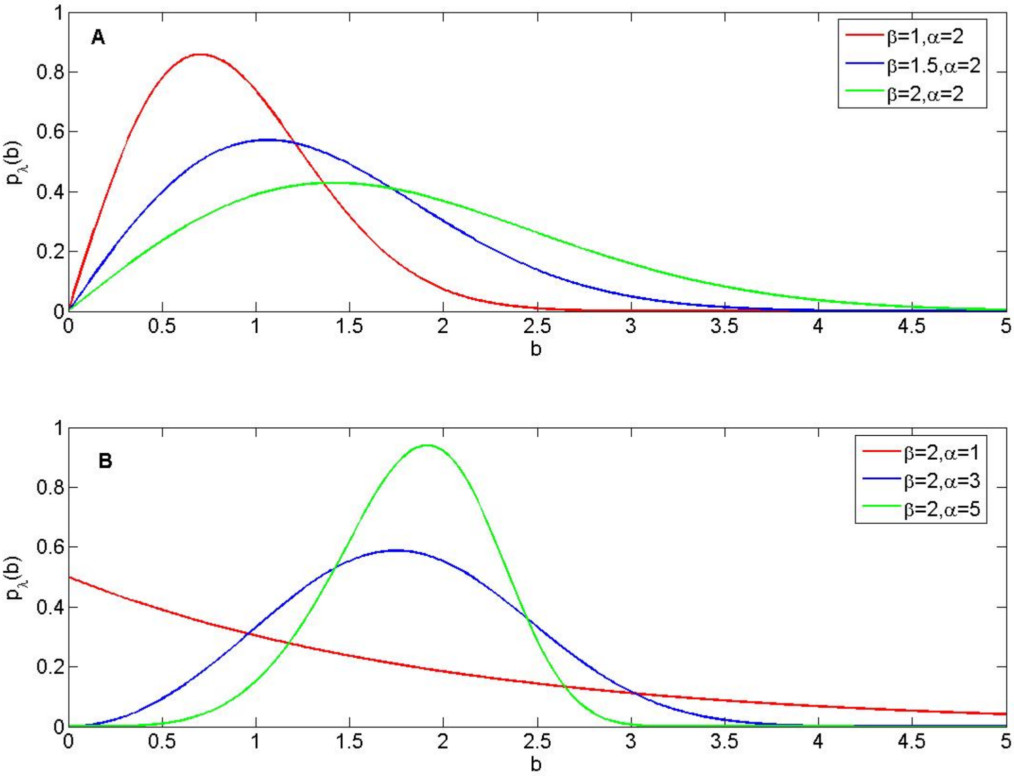

Fig. 1. Parameterization of the Weibull distributions. **A**: a scale parameter β; **B**: a shape parameter α.

The probability of the model solution $h(A \mid \Phi, M, H)$ is usually approximated by means of Monte Carlo simulations.

To relax the precision inherent in a probabilistic model, the single measure can be replaced by a **set of probability measures**

$$PM = \{p_\lambda : \lambda \in \Lambda\}.$$

The idea of a set of probability measures is inherent in Bayesian statistics, where the distribution parameters λ are considered to be random variables themselves. In the frequentist interpretation, statistical confidence regions induce parameterized families of distributions which in turn give raise to sets of probability measures [7].

A set of probability measures defines lower and upper probabilities according to the rules

$$\underline{P}(A \in S) = \inf\{P(A \in S) : P \in PM\}, \quad \overline{P}(A \in S) = \sup\{P(A \in S) : P \in PM\}.$$

If model uncertainties are represented through a set *PM* of probability measures, the model solutions form a set of probability measures as well. The corresponding lower and upper probabilities are computed by solving an optimization problem.

Random sets

Formally, the random set *RS*, also referred to as the Dempster-Shafer structure, is the set of all possible sub-sets B_i of the universal set *X* (the set of all states under consideration), including the empty set Ø. Each sub-set B_i is mapped into the interval [0,1] by means of the *m* function $m : RS(X) \to [0,1]$ satisfying two constrains: *m*(Ø)=*0*, $\sum_{B_i \in RS(X)} m(B_i) = 1$. A function *m* is called a **basic belief assignment** and its value is named a **basic belief mass (bbm)**. The lower and upper probabilities are defined by two non-additive continuous measures

$$\underline{P}(S) \equiv bel(S) \leq P(S) \leq pl(S) \equiv \overline{P}(S).$$

The **degree of belief** of *S*, *bel(S)*, determines the total amount of justified specific support given to *S*. It is obtained by summing all the basic belief masses of subsets B_i of *S*

$$bel(S) = \sum_{B_i \subset S} m(B_i).$$

We stated "justified" as *bel(S)* sums up only the basic belief masses given to subsets of *S*. The **degree of plausibility** of *S*, *pl(S)*, determines the maximum amount of potential specific support that could be given to *S*. It is defined as the sum all the basic belief masses of the sets B_i that intersect *S*

$$pl(S) = \sum_{B_i \cap S \neq 0} m(B_i).$$

We say "potential" as *pl(S)* includes the basic belief masses that could be transferred to non-empty subsets of *S* if some new information could justify such a transfer.

The two measures are related by $pl(S) = 1 - bel(\bar{S})$.

The combination (the **joint mass**) $m_{12} = m_1 \oplus m_2$ of two independent sets of mass assignments is determined by Dempster's rule of combination

$$m_{12}(S) = (m_1 \oplus m_2)(S) = \frac{\sum_{B \cap C = S \neq 0} m_1(B) m_2(C)}{1 - \sum_{B \cap C = 0} m_1(B) m_2(C)}.$$

Dempster-Shafer theory is a generalization of the Bayesian theory of subjective probability. The theory is based on two ideas: the idea of obtaining degrees of belief for one question from subjective probabilities for a related question, and Dempster's rule for combining such degrees of belief when they are based on independent items of evidence. These ideas can be illustrated by taking the examples published in [13,14].

Let us suppose that the subject **B** tells us that a tree fell on car. We assign to the subject **B** the subjective probability of 0.9 that **B** is reliable; our subjective probability that **B** is unreliable is 0.1. **B**'s statement is true if **B** is reliable, but it is not necessarily false if **B** is unreliable. Thus, we say that **B**'s testimony alone justifies a 0.9 degree of belief that a tree fell on car, but only a zero degree of belief (not a 0.1 degree of belief) that no tree fell on car. This zero does not mean that we are sure that no limb fell on car, as a zero probability does; it merely means that **B**'s testimony gives us no reason to believe that no tree fell on car.

To illustrate Dempster's rule for combining degrees of belief let us consider two independent witnesses by subjects **B** and **C** of the reliabilities P_1 and P_2, respectively. Using independence to compute joint probabilities, the probability that both are reliable is P_1P_2, the probability that neither is reliable is $(1-P_1)(1-P_2)$, and the probability that at least one is reliable is 1- $(1-P_1)(1-P_2)$. If they both claim that a tree fell on car, at least one of them is reliable, and hence we may assign this event a degree of belief of 1- $(1-P_1)(1-P_2)$.

Suppose, on the other hand, that **B** and **C** contradict each other: **B** says that a tree fell on car, and **C** says no tree fell on car. Their reliabilities are no longer subjectively independent for us. They can not be simultaneously right, only one is reliable, or neither is reliable. In such a case, our degree of belief in **B**'s testimony is defined as $P_{12} = \frac{P_1(1-P_2)}{1-P_1P_2}$, our degree of belief in **C**'s testimony is $P_{21} = \frac{P_2(1-P_1)}{1-P_1P_2}$. Hence we have a P_{12} degree of belief that a tree did fall on car (because **B** is reliable) and a P_{21} degree of belief that no tree fell on car (because **C** is reliable).

Pearl has argued [9,10] that it is misleading to interpret belief functions as representing either probabilities of an event, or the confidence one has in the probabilities assigned to various outcomes, or degrees of belief (or confidence, or trust) in a proposition, or degree of ignorance in a situation. Instead, according to Pearl, belief functions represent the probability that a given proposition is provable from a set of other propositions, to which probabilities are assigned.

Baudrit et al proposed the unified representation of incomplete probabilistic knowledge which can be encountered in risk evaluation problems [2]. Among others, proposed approach uses belief functions.

Fuzzy sets

Fuzzy sets can be interpreted as ordered families of sets or as membership functions [7]. Let B denotes a fuzzy real number. According to the first interpretation, B is considered as a family of parameterized intervals. The parameterization is determined by levels α; $\alpha \in [0,1]$. There corresponds to each α level an interval B^{α} so that $B^{\beta} \subset B^{\alpha}$ if $\alpha \leq \beta$. The intervals are nested and they can be characterized by their left/right contour functions (Fig. 2).

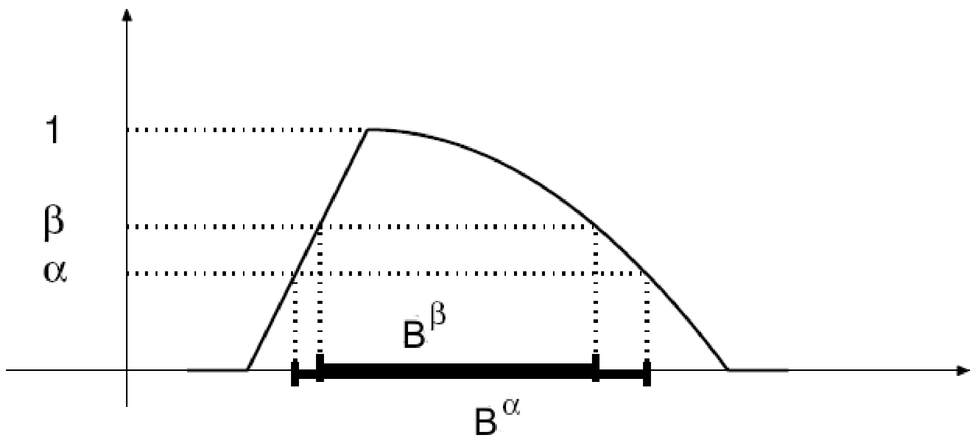

Fig. 2. Fuzzy set as ordered family of sets. Adopted from [7].

In the view of the second approach to fuzzy sets, a fuzzy set *B* is considered as a transformation of the real line to the interval [0,1], $\pi_B(b) \to [0,1]$, where b denotes a real number (Fig. 3). $\pi_B(b)$ is interpreted as the membership degree to which *b* belongs to the fuzzy set *B*, or as the degree of possibility that the scalar variable *B* takes the value *b*. The

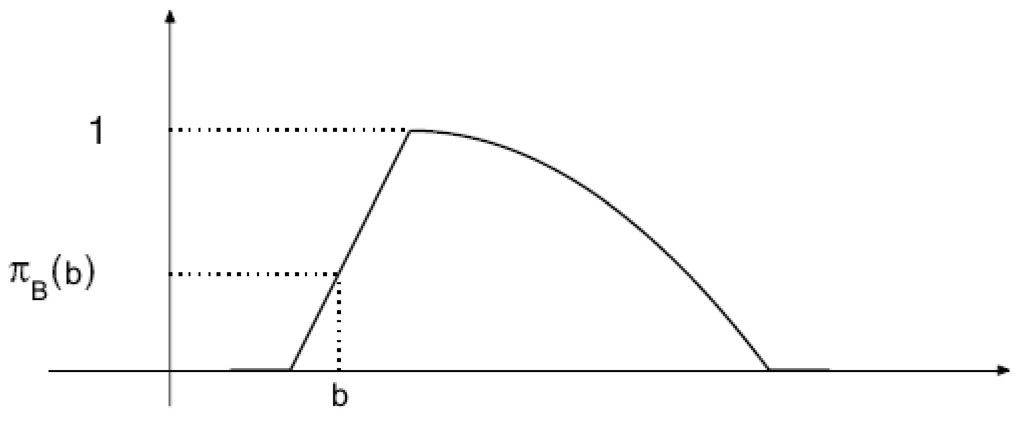

Fig. 3. Fuzzy set as degree of possibility. Adopted from [7].

intervals from the first interpretation are called *α*-level sets $B^{\alpha} = \{b : \pi_B(b) \geq \alpha\}$ [7].

It is possible to introduce a **possibility measure** on the underlying set $\pi_B(S) = \sup\{\pi_B(b) : b \in S\}$. It determines the degree of possibility that the parameter *B* takes a value in S. A possibility measure is in one-to-one correspondence with fuzzy sets; given a possibility measure π, its evaluation on singletons defines the membership function of a fuzzy set.

Oberguggenberger proposed the following procedure for constructing a fuzzy set [7]. At the beginning of the procedure the linguistic meaning of the α-values is specified verbally by the designing engineer and it remains fixed during the whole modeling process. For example, $\alpha=1$ denotes the standard value of B, $\alpha=2/3$ indicate high degree of possibility, $\alpha=1/3$ and $\alpha=0$ represent, respectively, medium, and low degree of possibility. The procedure starts by specifying the standard value b_s of B (deterministic approximation). The degree of possibility $\alpha=1$ is assigned to b_s. In the next step, possible deviations of B from the standard value are taken into account by gradually decreasing the degree of possibility until minimal and maximal possible values of B are reached at the level $\alpha=0$ (low possibility)(Fig. 4).

Another useful method allowing to determine a fuzzy set from a little sample of data can be found in [19].

The propagating of a fuzzy input B through a model, here formally denoted by a transformation $c=F(b)$, is determined by the **Zadeh extension principle** [18] given by

$$\pi_{F(B)}(c) = \sup\{\pi_B(b) : F(b) = c\}.$$

If the input consists of a vector of parameters $B = (B_1, \ldots, B_m)$, the principle takes the form

$$\pi_{F(B)}(c) = \sup\{\min(\pi_{B_1}(b_1), \ldots, \pi_{B_m}(b_m)) : F(b_1, \ldots, b_m) = c\}.$$

The Zadeh extension principle can be intuitively interpreted from the possibilistic point of view as the following. To determine the membership degree of the dependent variable $c=F(b_1,b_2)$, we should consider all possible combinations of (b_1,b_2) providing c. Each single combination is evaluated by the degree of possibility $\min(\pi_{B_1}(b_1), \pi_{B_2}(b_2))$ and the final result is given by the maximal degree of possibility (supremum) [4].

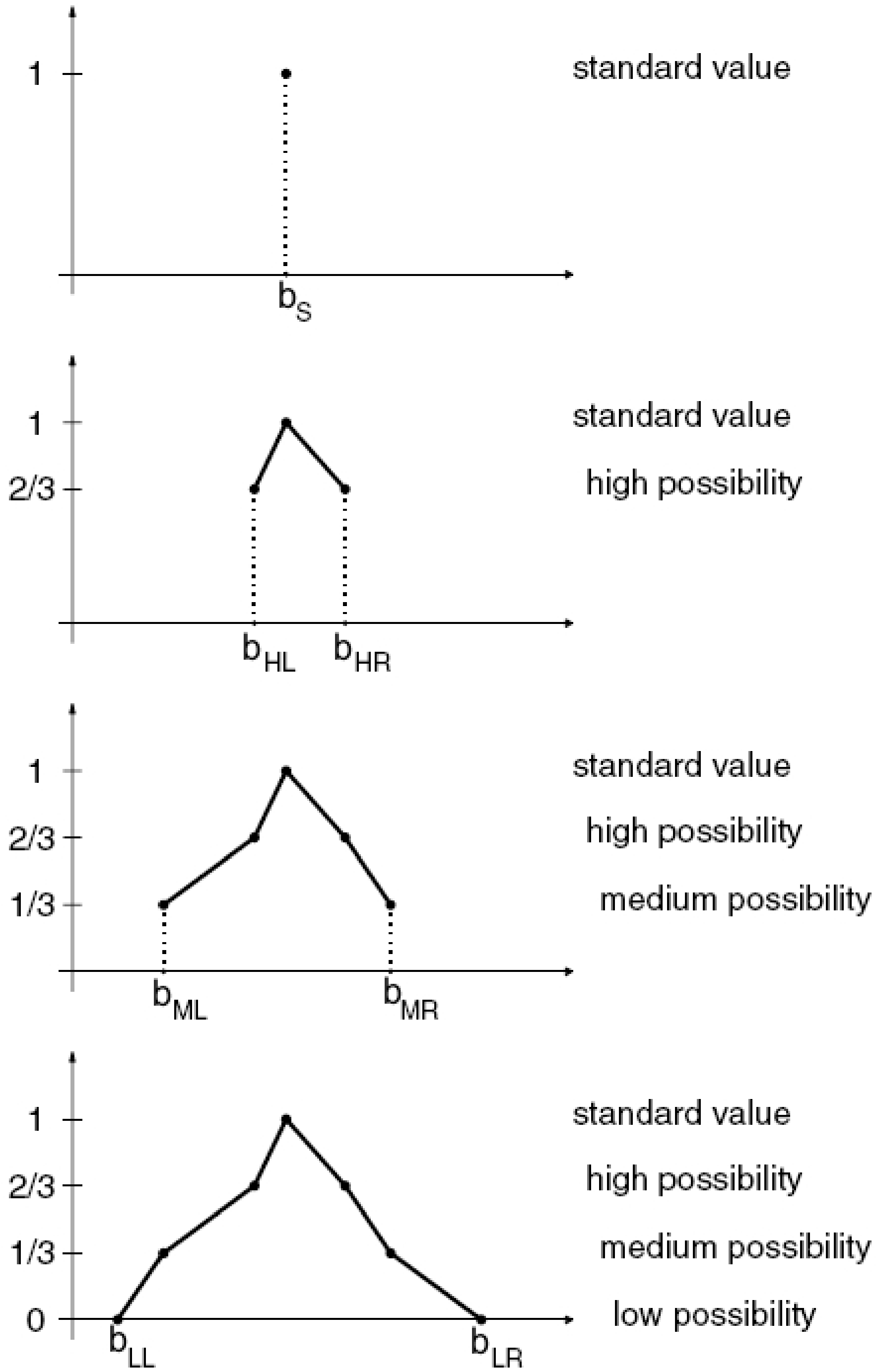

Fig. 4. Procedure for constructing a fuzzy set. Adopted from [7].

A variety of applications of fuzzy models in civil engineering developed by the research group of the University of Innsbruck is reviewed in [4].

In practice, it may occur that certain model parameters can be reasonably represented by probability distributions, while others are better represented by fuzzy numbers due to data scarcity. Guyonnet et al [5] proposed a hybrid approach which combines Monte Carlo sampling of probability distribution functions with fuzzy calculus. The approach was applied to a real case of estimation of human exposure to cadmium present in the soils located in the north of France.

An extensive list of references on approaches briefly introduced in this section can be found in [7].

3. The transferable belief model

The transferable belief model (TBM) is intended to represent quantified beliefs based on belief functions. Within the TBM, beliefs are quantified at two levels: 1) a **credal level** where beliefs are entertained and quantified by belief functions, 2) a **pignistic level** where beliefs are used to make decisions and are quantified by probability functions [16, 17].

The TBM postulates 1) the existence of the **bbm** and 2) the **bbm** given to a set *S* is transferred to subsets of *S* when new information becomes available. All other properties of the TBM are derived from very general principle [16].

Although the TBM shares the same concepts as considered by a generalization of the Bayesian model, a random sets model, or an upper and lower probability model, the TBM relies on its own interpretation of the Dempster-Shafer model.

The set of belief functions at the credal level are transformed to the set of probability functions at the pignistic level by means of the **pignistic transformation**

$$BetP(S) = \sum_{C \in \Re} m(C) \frac{|S \cap C|}{|C|},$$

where $\Re$ denotes the Boolean algebra of the subsets of the universe of discourse.

The TBM is supplied with several tools such as a rule of minimal commitment, a cautious rule of (conjunctive) combination, a rule of discounting weighted beliefs received from a partially reliable source, etc. Those concepts are perfectly justified within the framework of the TBM, although many of them are often erroneously considered as arbitrary [16].

According to **the principle of minimal commitment**, when several belief functions are compatible with our knowledge, we should select the one that gives the minimum support to every propositions (when possible). The selected belief function is called the least committed. The principle corresponds to: 'don't give more support than justify'. The principle will be illustrated by considering a simple expert judgment problem of assessing an unknown probability *P* [16]. Let us suppose that the expert is asked to bet that *P* is in some intervals of [0,1]. And assume that the data collected from the expert will be some percentiles x_p of the meta-probability distribution about *P.* The meta-probability function corresponds to the pignistic probability function $BetP(P \leq x_p) = p$ and thus the problem is to find the belief function which pignistic transformation satisfies the known constraints, i.e. $BetP(P \leq x_p)$. Suppose the collected percentiles similar as in [16]: $x_{0.05}$=0.5, $x_{0.5}$=0.7 and $x_{0.95}$=0.8.

The first step is to determine **bbm** corresponding to the whole interval [0,1]. It should be compatible with *BetP([0,0.5])=0.05* and it is spread equally on the interval [0,1] by the pignistic transformation. These constrains are satisfied by *m([0,1])=0.1*. The next step is to account for the constraint imposed to the interval [0.8,1] by $x_{0.95}$ which determines the pignistic probability of [0.8,1] as *BetP([0.8,1])=0.05*. The [0.8,1] interval receives a probability of *m([0,1])(1-0.8)=0.02* from the interval [0,1]. The missing piece of the pignistic probability, *BetP([0.8,1])-0.02=0.03,* is assigned to the largest remaining interval [0.5, 1]. Knowing that the portion of [0.5,1] which contributes to [0.8,1] is 2/5, it is readily found that *m([0.5,1])=0.03*5/2=0.075*. By using similar procedure, Smets derived **bbms** of the remaining intervals. The *m*-values are *m([0.5,0.8])=0.6* and *m([0.5,0.7])=0.225*.

If we know a priori the reliability of the expert, the data provided by the expert must be reduced as they are not fully reliable. Let α denotes the strength of the reliability we give to the expert knowledge. Then, according to the **discounting principle** [12], all degree of belief *bel(S)* are reduced by a factor α and the amount of **bbm** lost by this process is reallocated to the whole interval [0,1]:

$$bel^{\alpha}(S) = \alpha bel(S),\ for\ all\ S \in \Re,\ S \neq [0,1]\ .$$

A value $\alpha=1$ denotes a full reliability, $\alpha=0$ represents a total unreliability.

Let us assign the reliability of $\alpha=0.8$ to the expert from the above example. Then the belief function bel^{α} is characterized by the following **bbm**: $m^{\alpha}([0,1]) = 0.28$, $m^{\alpha}([0.5,1]) = 0.06$, $m^{\alpha}([0.5,0.8]) = 0.48$, $m^{\alpha}([0.5,0.7]) = 0.18$.

The interested reader is referred to the following internet sources:

- publications on the **TBM** by Smets et al:
  http://iridia.ulb.ac.be/~psmets/AABPapers.html
- software implementation of the **TBM** (written in MATLAB):
  http://iridia.ulb.ac.be/~psmets/#G .

4. Examples

This section exemplifies the power of subjective approaches to treating uncertainty.

4.1. The murder of Mr. Jones

To show the importance of deciding which of the two concurrent models, the TBM or the Bayesian, could be best, we investigate the problem of Mr. Jones's murder [14]. It will be shown that the two analysis lead to diametrically opposed conclusions. The choice between the two models is thus an important issue.

4.1.1. The problem

I am a judge analyzing the Mr. Jones's case. I know that Mr. Jones was murdered by one of the three people whose names are Peter, Paul and Mary (evidence E_0). I also know that the killer was selected by a throw of a dice. The applied rule was: if it is an even number, the killer will be female, if it is an odd number, the killer will be male (evidence E_1). But I do not know: I) what the outcome was, and II) in the case of an odd number, how it would be decided between Peter and Paul. Thus, my bet supported by the available information (evidences E_0, E_1) is 1 to 1 for male versus female.

Afterwards, I learn that Peter was not selected. He went to the police station at the time of the killing in order to have a perfect alibi (evidence E_2). How I should bet now on male versus female? The applications of the TBM and the Bayesian model lead to significantly different answers. The TBM approach suggests maintaining the bet 1 to 1, the Bayesian model supports the bet 1 to 2. The detailed analysis of the problem follows for each approach separately.

4.1.2. The TBM analysis

The evidence E_0 and its basic belief assignment m_0 are:

$$E_0: \quad k \in \Omega = \{Peter, Paul, Mary\}, \quad \Re_0 = 2^{\Omega}$$

$$m_0(\{Peter, Paul, Mary\}) = 1,$$

where k denotes the killer.

The dice throwing experiment (evidence E_1) gives belief assignment m_1:

$$E_1: \quad \Re_1 = \{Female, Male\}$$

$$m_1(Female) = 0.5, \; m_1(Male) = 0.5 .$$

Conditioning m_0 on E_1 by Dempster's rule of conditioning induces m_{01}:

$$E_{01}: \quad E_0 \text{ and } E_1, \quad \Re_{01} = 2^{\Omega}$$

$$m_{01}(\{\text{Mary}\}) = 0.5, \; m_{01}(\{\text{Peter, Paul}\}) = 0.5 .$$

The evidence E_2 (Peter's alibi) gives m_2:

$$E_2: \quad k \in \Omega = \{Paul, Mary\}, \; \Re_2 = 2^{\Omega}$$

$$m_2(\{Paul, Mary\}) = 1.$$

Conditioning m_{01} on E_2 leads to m_{012}:

E_{012} : E_{01} and E_2, $\Re_{012} = 2^{\Omega}$

$$m_{012}(\{\text{Mary}\}) = 0.5 \,,\; m_{012}(\{\text{Paul}\}) = 0.5 \,.$$

The basic belief mass that was given to {Peter or Paul} is transferred to {Paul}. My bet on female versus male would be 1 to 1, as before obtaining the evidence E_2.

4.1.3. The Bayesian analysis

In the Bayesian approach, my degrees of belief are quantified by probability distributions and all pieces of evidence are incorporated through the Bayesian conditioning processes. A probability distribution P_1 on Ω = {Peter, Paul, Mary} corresponding to E_1 is:

$$P_1(k \in \{Mary\}) = 0.5 \,,\; P_1(k \in \{Peter, Paul\}) = 0.5 \,.$$

My bet on female versus male would also be 1 to 1.

After learning the evidence E_2, I would compute P_{12} as:

$$P_{12}(k \in \{Mary\}) = P_1(k \in \{Mary\} \mid k \in \{Mary, Paul\}) = \frac{P_1(k \in \{Mary\})}{P_1(k \in \{Mary\}) + P_1(k \in \{Paul\})} = \frac{0.5}{0.5 + x} \,,$$

where x denotes an unknown probability $P_1(k \in \{Paul\})$. The application of either the insufficient reason principle or a symmetry argument or a minimum entropy argument leads to x = 0.25. Although this is the most natural assumption, any other value from the interval [0, .5] can be taken in consideration. Such value would correspond to some a priori probability on Peter versus Paul, the piece of the information, which is not supported by any of the available pieces of evidence.

4.2. The meaning of the failure probability in a (simple) geotechnical design problem

Using a simple geotechnical design problem Oberguggenberger [8] showed that the failure probability may depend in an extremely sensitive way on the choice of distribution function. They concluded that in such case the failure probability has no meaning as a frequency of failure. They suggest more robust alternatives, as interval probability or fuzzy sets. The work unambiguously exemplifies the power of these alternatives, and it is summarized in this section.

4.2.1. The probabilistic safety concept

Let R and S are two groups of random variables, R a group of all variables describing the resistance of investigated structure, while S a group of variables describing the loads. An engineering model provides the limit state function g(R,S) whose negative values indicate the failure. Knowing the probability distributions of R and S, we can compute the failure probability

$$p_f = P(g(R,S) < 0).$$

However, in practice, using the current codes the designing engineer has to verify a relation of the type

$$R_k / \gamma_R \geq \gamma_S S_k,$$

where R_k and S_k denote certain percentiles of R and S (critical values) and γ_R and γ_S are partial safety factors, which are usually in the codes prescribed.

4.2.2. Fitting probability distributions to a soil parameter

The characteristic value of a soil parameter is the shear strength of the soil $\tau_f = c + \sigma \tan\varphi$, where c denotes the cohesion and σ the normal stress. Oberguggenberger [8] analyzed distribution of the measured friction coefficient $\upsilon = \tan\varphi$. Angles φ were obtained from twenty direct shear tests. They tested reliability of four distributions: normal distribution, lognormal distribution with two parameters, lognormal distribution with three parameters, and triangular distribution. As can be seen from Fig. 5**A**, the former two distributions do not mimic the peak asymmetry. A better fit is achieved by the later two distributions (Fig. 5**B**). All four fitted distributions were tested by means of the Kolmogorov-Smirnov test and of χ^2 test. The goodness of fit was found reasonable in all cases.

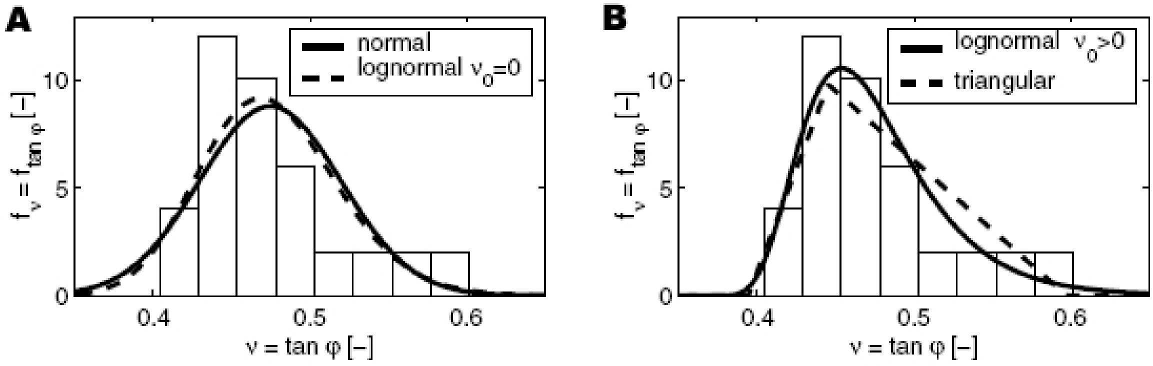

Fig. 5. Fitting probability density functions to friction angle data. **A**: normal and two parameter lognormal distribution; **B**: three parameter lognormal and triangular distribution. Adopted from [8].

4.2.3. A centrically loaded square footing

As an example, Oberguggenberger [8] investigated a problem of centrically loaded square footing (Fig. 6).

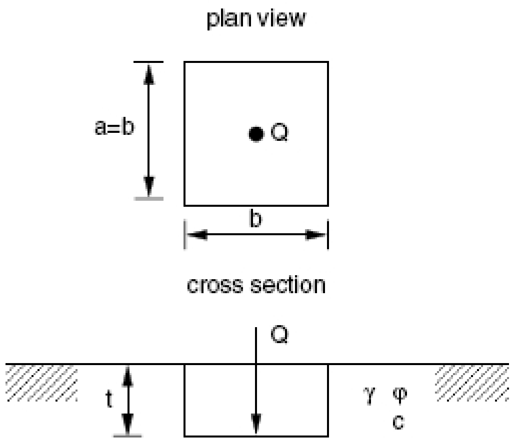

Fig. 6. A centrically loaded square footing. Adopted from [8].

They performed a series of numerical simulations. In each simulation, they calculated N independent realizations of the friction coefficient $\nu = \tan\varphi$ and of the load S. The characteristic load $S_k = Q_{f,d} / \gamma_S$ was interpreted as 95%-percentile of a normally distributed load, i.e. $S_k = \mu_S + k_N \sigma_S$. For each realization of the friction coefficient ν, there was calculated corresponding realization of the resistance R [8, Eq. 3]. The histograms of z= r−s for four considered distributions are shown in Fig. 7. The failure probability p_f was estimated from the relative frequency of failure events Z<0, i.e. $p_f \approx H_f / N$, where H_f denotes the

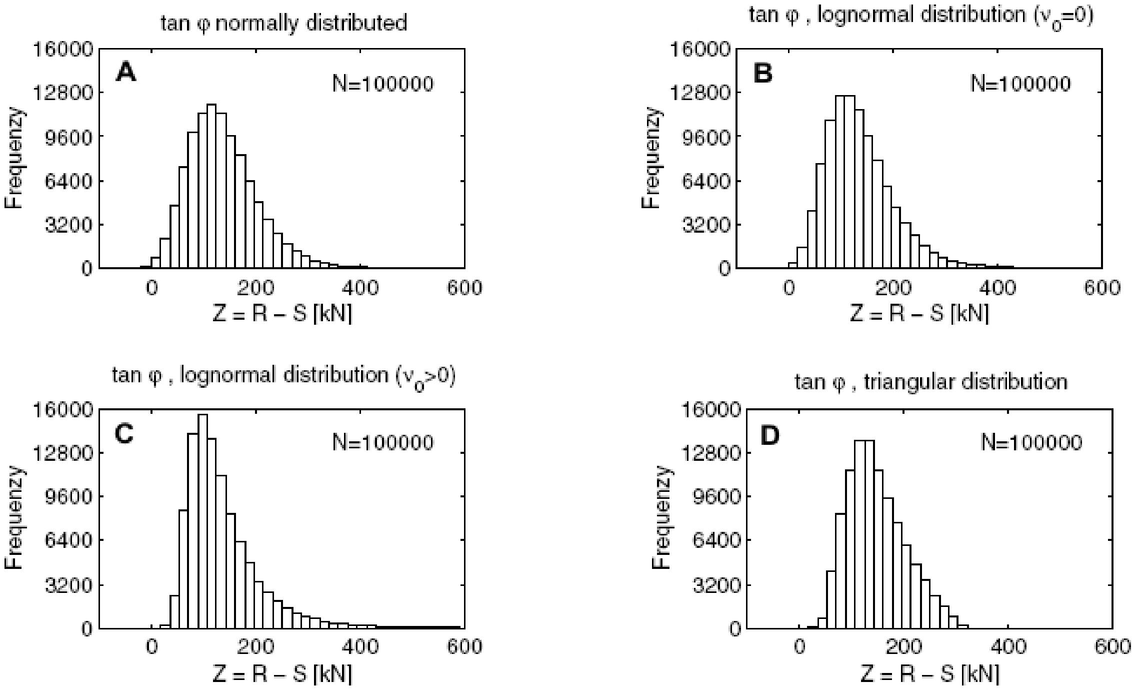

Fig. 7. Simulated histograms of z=r-s for **A**: normal distribution; **B**: two parameter lognormal distribution; **C**: three parameter lognormal distribution; **D**: triangular distribution. Adopted from [8].

number of failure events. The estimated failure probabilities vary in between 10^{-3} and 10^{-11} for different distributions, (Tab. 1). Such huge variations of several orders of magnitude were confirmed by analytical computations. The failure probability is highly sensitive to the shape of distribution, it is not a robust measure.

Table 1. Failure probabilities p_f in *m* simulations with *N* realizations. Adopted from [8].

Distribution	p_f	N	m
Normal	0.81×10^{-3}	10^8	4
Two parameter lognormal	1.1×10^{-4}	10^8	4
Three parameter lognormal	1.0×10^{-9}	10^{11}	5
Triangular	1.0×10^{-11}	10^{11}	9

4.2.4. Robust alternatives

Oberguggenberger [8] provided following three alternatives.

4.2.4.A Set of probability measures

Oberguggenberger [8] calculated the lower and upper failure probabilities considering all probability distributions that could feasibly produce the data in the interval $[\underline{\varphi},\overline{\varphi}]=[20^o,30^o]$. They obtained:

$$\underline{p}_f=\inf\{P(Q_f(\varphi)<S):\varphi\in[20^o,32^o]\}=P(S>Q_f(\overline{\varphi}))=0\,,$$

$$\overline{p}_f=\sup\{P(Q_f(\varphi)<S):\varphi\in[20^o,32^o]\}=P(S>Q_f(\underline{\varphi}))=0.00235\,.$$

Less conservative estimates would be obtained by restricting the set of admitted distributions.

4.2.4.B Random sets

The parameters of the probability distributions were considered to be random. The randomness was modeled by means of random sets. As a probabilistic modeling approach would just transfer the difficulties with the choice of type of distribution to the second level.

The load was assumed normally distributed. It was postulated, that the friction coefficient ν has a normal distribution. This assumption was applied to make easy the calculation of confidence intervals $\lfloor l_{q_i}, l_{q_k} \rfloor$, which were calculated via

$$l_q=\overline{\upsilon}-\frac{t_q s_\nu}{\sqrt{n}}\ ,$$

where $\overline{\upsilon}=4.74$ is the sample mean value, t_q the q.100%-percentile of t-distribution, $s_\nu=0.0452$ the sample standard deviation, and n=20 the sample size. Obtained confidence intervals are displayed in Fig. 8.

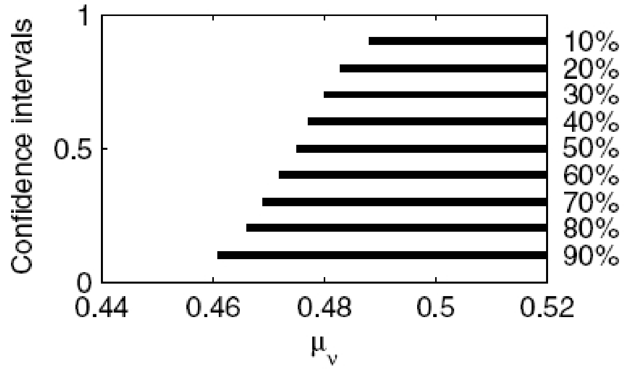

Fig. 8. Confidence intervals for μ_ν. Adopted from [8].

This enables to construct 10 focal sets A_i, each with the basic probability weight $m_i = 0.1$ (Tab. 2). Finally, each A_i was mapped into a corresponding interval $B_i = [\inf\{p_f(\mu_v) : \mu_v \in A_i\}, \sup\{p_f(\mu_v) : \mu_v \in A_i\}]$ of failure probabilities (Tab. 2).

Plausibilities of the events $\mu_v \geq l$ and $p_f \leq p$ are shown in Fig. 9.

Table 2. Focal sets. Adopted from [8].

Weight	A_i	B_i
0.1	(−∞, 0.461]	[0.00205, 1]
0.1	(0.461, 0.466]	[0.00150, 0.00205)
0.1	(0.466, 0.469]	[0.00119, 0.00150)
0.1	(0.469, 0.472]	[0.00098, 0.00119)
0.1	(0.472, 0.475]	[0.00081, 0.00098)
0.1	(0.475, 0.477]	[0.00067, 0.00081)
0.1	(0.477, 0.480]	[0.00055, 0.00067)
0.1	(0.480, 0.483]	[0.00043, 0.00055)
0.1	(0.483, 0.488]	[0.00030, 0.00043)
0.1	(0.488,∞]	[0, 0.00030)

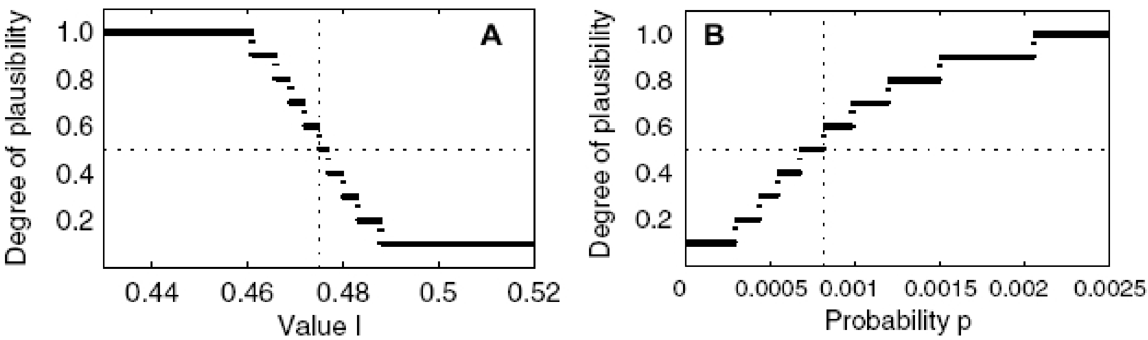

Fig. 9. Plausibility of **A**: event $\mu_v \geq l$; **B**: event $p_f \leq p$. Adopted from [8].

4.2.4.C Fuzzy sets

The fuzzy set describing randomness of ν can be constructed using an expert's risk assessment of possible ranges of the angle φ. The experts were asked to determine I) the range of ν which has the degree of possibility α = 0.5; and II) the value ν which has the possibility α = 1. Fig. 10A. shows resulting triangular fuzzy number. The fuzzy set describing the resistance was computed by applying the rules of fuzzy set theory and is depicted in Fig. 10B.

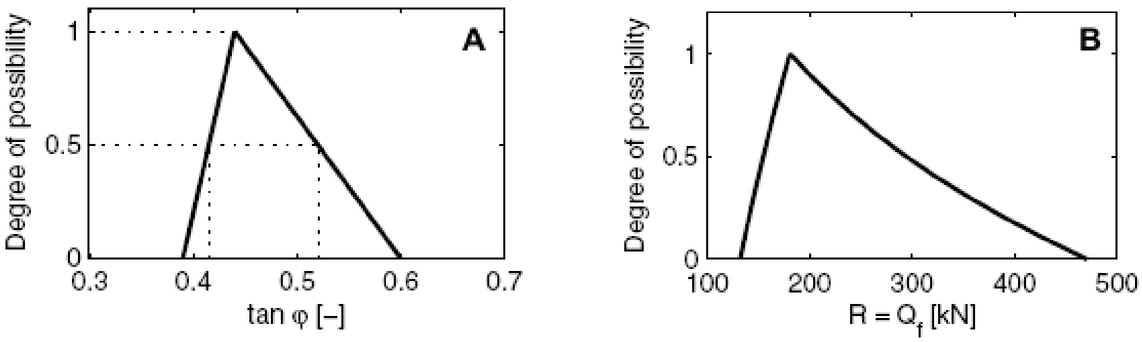

Fig. 10. A: Constructed triangular fuzzy number for ν=tan φ. **B**: Computed fuzzy number for the resistance R. Adopted from [8].

5. Discussion

PA often requires the investigation of the consequence of rare event for which only few data are available. The application of the probability model of uncertainty may suffer from lack of information. The probabilistic model would become random itself in such case. As the result, parameterized families of distributions give raise to sets of probability measures. The basic from several approaches to construct a set of parameterized measures was briefly reviewed. It turns out that the subjective theories of probability are well suited for this task; in fact, they are designed to it. The subjective theories allow for a formalization of vague data as well as for a possibility theoretic interpretation of computation results. The probability of the event is replaced by the degree of belief in the particular scenario of the event.

SWOT Analysis

- **S**trengths: to treat uncertainties of rare event formally, within a mathematical structure.
- **W**eaknesses: more suitable for qualitative reasoning than for quantitative estimation of uncertainty.
- **O**pportunities: the attempt to incorporate suitable subjective probability concepts into PA may be considered as the research challenge within PAMINA.

- Threats: the numerical simulations may demand for an excessive computation effort.

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